AD 746466

ONR Scientific Report
Contract Report No. 18

20

DEPARTMENT
OF
APPLIED
MECHANICS

Variational Principles in Dynamic Thermoviscoelasticity

by

Subrata Mukherjee

STANFORD
UNIVERSITY
STANFORD,
CALIFORNIA
94305

Office of Naval Research
Grant N00014-67-A-0112-0060

SUDAM Report No. 72-3

APRIL 1972

40

VARIATIONAL PRINCIPLES
IN
DYNAMIC THERMOVISCOELASTICITY

Subrata Mukherjee
Department of Applied Mechanics
Stanford University
Stanford, California 94305*

*This research was supported by Contract No. N00014-67-A-0112-0060 of the Office of Naval Research, Washington, D.C.

unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

Stanford University
Department of Applied Mechanics
Stanford, California 94305

2a. REPORT SECURITY CLASSIFICATION

unclassified

2b. GROUP

-

3. REPORT TITLE

VARIATIONAL PRINCIPLES IN DYNAMIC THERMOVISCOELASTICITY

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Interim

5. AUTHOR(S) (First name, middle initial, last name)

SUBRATA MUKHERJEE

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
APRIL 1972	38	14

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
N00014-67-A-0112-0060	SUDAM REPORT NO. 72-3
b. PROJECT NO. NR 064-471	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	Contract No. 18

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	OFFICE OF NAVAL RESEARCH DEPARTMENT OF THE NAVY WASHINGTON, D. C. 20025

13. ABSTRACT

Dual variational principles for steady state wave propagation in three dimensional thermoviscoelastic media are presented. The first one, for the equations of motion, involves only the complex displacement function. The second principle is for the energy equation. The specialized versions of these principles in two-dimensional polar coordinates and then in one dimension are obtained. A one-dimensional example, that of wave propagation in a thermoviscoelastic rod insulated on its lateral surface and driven by a sinusoidal stress at one end, is solved using the Rayleigh-Ritz method. The displacement and temperature functions are expressed as series of polynomials. Successive approximations for the solution are compared with a solution obtained by a method of finite differences, and an estimate of the degree of accuracy as a function of the number of terms taken in the series is obtained. It is found that as long as the spatial distribution of stress and temperature are sufficiently smooth, r[illegible] convergence to the correct solution is obtained. If the stress is a rapidly oscil[illegible]ting function of the distance along the rod, polynomials are no longer efficient and other test functions must be chosen.

DD FORM 1 NOV 65 1473

S/N 0101-807-6801

unclassified

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
viscoelasticity thermal coupling variational principles Rayleigh-Ritz method						

unclassified

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
viscoelasticity thermal coupling variational principles Rayleigh-Ritz method						

DD FORM 1 NOV 65 1473 (BACK)
(PAGE 2)

unclassified

Security Classification

INTRODUCTION

Several authors have developed and used variational principles to obtain solutions to problems in quasi-static and dynamic viscoelasticity, with and without thermomechanical coupling. Gurtin [1]* and Leitman [2] have developed variational principles for viscoelastic media without thermomechanical coupling. They have used the convolution form of the constitutive equations and have developed variational principles for several types of boundary value problems. Their work appears to be primarily of mathematical interest. Valanis [3] has developed a principle applicable to viscoelastic materials with constant Poisson's ratio, without thermomechanical coupling.

Schapery [6,7] has studied wave propagation in viscoelastic media with thermomechanical coupling. In [7] he has used a complex modulus form of the constitutive equations and has developed a variational principle analogous to Reissner's complementary principle using complex kinetic and potential "energy" functions. His principle, however, involves both stress and displacement functions which must already satisfy the equations of motion. He has considered examples with bodies that are either massless or with concentrated mass, and in his last example of a 'solid cylinder with distributed mass' he only gives a first approximation to the solution, using only one term of a series expansion. While his method appears promising, the question of convergence to the exact solution, or, in other words, how many terms in the series are necessary to get a sufficiently close approximation to the exact solution, remains open.

*Numbers in square brackets designate references at the end of this report.

This report is concerned with the application of variational principles to problems of steady state wave propagation in viscoelastic media with thermomechanical coupling. A complex modulus description of the constitutive equations is used. The material is assumed to be thermorheologically simple [5] and the energy equation, as suggested by Schapery [7], uses the cycle averaged temperature distribution with the cycle averaged value of the Rayleigh dissipation function acting as the heat source. The displacement variational principle suggested here involves only the complex displacement function and an admissible set of displacement functions need only satisfy any prescribed displacement boundary conditions that might exist. This principle can be considered to be an extension of that developed by Kohn, Krumhansl and Lee [4] for elastic media. It uses complex instead of real "energy" functions. The temperature variational principle is the one suggested by Biot [8] and Schapery [7].

An alternative form of these principles is suggested. This proves more useful for certain applications. These principles are set up for general three-dimensional problems and are later specialized to the cases of two and one dimension.

As an example, the problem of steady state longitudinal waves in a viscoelastic rod with thermal coupling subjected to a sinusoidal stress applied at one end, is solved using a variational approach. Huang and Lee [9] solved this problem including time as an independent variable. This resulted in partial differential equations which were solved numerically using a method of finite differences. This is useful if the time histories of the stress and temperature have to be determined.

For most engineering design applications, however, the steady state values of stress and temperature are of primary interest, since due to dissipation of mechanical energy the temperature increases until a steady state is reached, if in fact the situation is stable. Such a steady state yields the most severe temperature conditions which are the major concern in design. In such cases it is simpler and far more efficient to obtain the steady state values directly instead of following the complete time history of the process till a steady state is reached. In this example, the steady state values of stress and temperature have been directly obtained by using a Rayleigh-Ritz procedure on the alternative form of the variational principles. Functions for displacement (complex) and temperature are assumed as polynomial series (for convenience) with "n" and "m" terms respectively, with unknown coefficients. Simultaneous extremization of two functionals is carried out by solving the resultant nonlinear algebraic equations in a computer. The number of terms "n" and "m" can be set in the program. Calculations for a Lockheed solid propellant [9] are carried out for various values for "n" and "m" and the question of rapidity of convergence to the solution given in [9] is discussed.

GOVERNING DIFFERENTIAL EQUATIONS

1. General Equations in Three Dimensions

Let us consider the governing differential equations for stresses, displacements and temperature in steady state oscillations of linear isotropic viscoelastic media. The thermomechanical coupling is caused by the cycle averaged value of the mechanical dissipation function acting as the heat source in the energy equation and by the fact that the complex viscoelastic modulii are temperature dependent. As pointed out by Schapery [6,7] the coupling terms due to the dilatation and potential energy drop out of the energy equation if it is integrated over a cycle.

We assume steady state conditions where the mechanical variables are harmonic functions of time, and the temperature, after a sufficiently long time, is independent of time. Strictly speaking, the temperature is never truly time independent but has small cyclic variations about a mean value as a result of the cyclic variations of the potential energy, dilatation and dissipation (see [9]). These small fluctuations, however, will be neglected and henceforth the temperature will mean its cycle averaged steady state value.

Let the stress and strain tensors and the displacement vector be defined as the real parts of

$$\begin{aligned} \tilde{\sigma}_{ij} &= \sigma_{ij}\, e^{i\omega t} \\ \tilde{\epsilon}_{ij} &= \epsilon_{ij}\, e^{i\omega t} \qquad (1) \\ \tilde{u}_i &= u_i\, e^{i\omega t} \end{aligned}$$

where $i = \sqrt{-1}$, ω is the frequency (real) and t is time. The complex quantities σ_{ij}, ϵ_{ij} and u_i are used most of the time in further calculations and will be referred to as simply stress, strain and displacement respectively. The familiar cartesian tensor notation is used here. The suffixes i and j range from 1 to 3 and a repeated index implies summation over that index.

The equations of motion are

$$\sigma_{ij,j} + \rho\, \omega^2\, u_i = 0 \tag{2}$$

where ρ is the mass density (assumed constant).

The constitutive equations, using the familiar complex modulus formulation, are

$$\sigma_{ij} = \lambda^* u_{k,k}\, \delta_{ij} + \mu^*(u_{i,j} + u_{j,i}) \tag{3}$$

where λ^* and μ^* are complex Lamé parameters which are functions of temperature, and δ_{ij} is the Kronicker delta.

Equation (3) is entirely analogous to the constitutive equations for elastic solids and is obtained directly by the use of the well known correspondence principle [12].

We note that Eqs. (3) should contain a term due to thermal expansion. However, as pointed out by Schapery in [7], in our problem each of the mechanical variables can be separated into two parts: (a) that due to applied cyclic loading in the absence of thermal expansion, and (b) that due to thermal expansion with homogeneous mechanical boundary conditions and a temperature distribution obtained from the components of (a). Here we are only concerned with the former components and the latter can be obtained from standard thermoelastic analysis.

The Lamé parameters are related to the more commonly used complex shear and bulk compliances J^* and B^* by the relations

$$\mu^* = \frac{1}{J^*}$$
$$\lambda^* = \frac{1}{B^*} - \frac{2}{3J^*} \tag{4}$$

Typically, in polymers, J^* is a very strong nonlinear function of temperature while B^* is a relatively weak function of temperature.

In thermorheologically simple materials it is assumed that J^* is a function of only the reduced frequency ω' which is related to the actual frequency ω through the temperature dependent shift factor a_T (see [7]), i.e.

$$\omega' = \omega \, a_T(T) \tag{5}$$

where a_T represents the effect of temperature on viscosity.

Combining Eqs. (2) and (3) one can write the equations of motion in terms of displacement alone

$$(\lambda^* u_{k,k})_{,i} + (\mu^* u_{i,j})_{,j} + (\mu^* u_{j,i})_{,j} + \rho\, \omega^2 u_i = 0 \tag{6}$$

The steady state energy equation for the cycle averaged temperature distribution is given by

$$KT_{,ii} = -2D \tag{7}$$

where T is the temperature, K is the thermal conductivity (assumed constant) and D is the cycle averaged value of the Rayleigh dissipation function given by

$$D = \frac{\omega}{4\pi} \int_t^{t+\frac{2\pi}{\omega}} \mathrm{Re}(\tilde{\sigma}_{ij})\, \mathrm{Re}\left(\frac{\partial \tilde{\epsilon}_{ij}}{\partial t'}\right) dt'$$

where Re denotes the real part of the complex argument. 2D is the cycle averaged value of the mechanical dissipation.

Using Eqs. (1), (3) and (7) and after carrying out the necessary integration, we have

$$D = \frac{\omega}{4}\left[\lambda_2|\epsilon_{kk}|^2 + 2\mu_2|\epsilon_{ij}|^2\right] \tag{8}$$

where

$$\lambda^* = \lambda_1 + i\lambda_2$$

$$\mu^* = \mu_1 + i\mu_2$$

λ_1 and μ_1 are the storage Lamé parameters and λ_2 , μ_2 the loss parameters,

$$|\epsilon_{kk}|^2 = \epsilon_{kk}\,\bar{\epsilon}_{jj}$$

$$|\epsilon_{ij}|^2 = \epsilon_{ij}\,\bar{\epsilon}_{ij}$$

and "‾" denotes the complex conjugate.

Note that as defined D is a real function of the strain tensor and the loss parameters λ_2 and μ_2 .

Since λ_2 and μ_2 are nonlinear functions of temperature, (7) is a nonlinear partial differential equation for the temperature. Also, the steady state temperature is purely a spatial function independent of time.

Using the familiar kinematic relations

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{9}$$

we can write Eq. (7) in terms of displacement and temperature

$$K\,T,_{ii} + \frac{\omega}{2}\left[\lambda_2|u_{k,k}|^2 + \mu_2(u_{i,j}\,\bar{u}_{i,j} + u_{i,j}\,\bar{u}_{j,i})\right] = 0 \tag{10}$$

Equations (6) and (10) are a complete set of four nonlinear partial differential equations for the four unknowns u_i $(i=1,3)$ and T. Since these are written in terms of displacements, the compatibility conditions are automatically satisfied.

For the boundary conditions we assume that the displacement vector is prescribed on a part of the surface A_u, while the traction vector is prescribed on the remainder A_σ. Also, the temperature is prescribed on the portion of the surface A_T and the heat flux (per unit area) is prescribed on the remaining surface A_H.

2. Equations in Two-Dimensional Polar Coordinates

It is instructive to look at the form of the general equations for the case of plane strain in polar coordinates. These equations and the associated variational principles to be discussed later are useful for problems involving long circular cylinders in plane strain.

The equations of motion now take the well known form

$$\begin{aligned} &\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + \rho\,\omega^2 u_r = 0 \\ &\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \rho\,\omega^2 u_\theta = 0 \end{aligned} \tag{11}$$

while the constitutive euqations become

$$\begin{aligned} \sigma_r &= \lambda^*(\epsilon_r + \epsilon_\theta) + 2\mu^*\,\epsilon_r \\ \sigma_\theta &= \lambda^*(\epsilon_r + \epsilon_\theta) + 2\mu^*\,\epsilon_\theta \\ \sigma_{r\theta} &= 2\mu^*\,\epsilon_{r\theta} \end{aligned} \tag{12}$$

where σ_r, σ_θ, $\sigma_{r\theta}$ are the stress components, ϵ_r, ϵ_θ, $\epsilon_{r\theta}$ the strain components and u_r , u_θ the displacement components in polar (r,θ) coordinates.

The strains and displacements are related by the familiar kinematic equations

$$\epsilon_r = \frac{\partial u_r}{\partial r}$$

$$\epsilon_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \tag{13}$$

$$\epsilon_{r\theta} = \frac{1}{2}\left\lfloor \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right\rfloor$$

The energy equation (Eq. (7)) takes the form

$$K\,\nabla^2 T = -2D \tag{14}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$D = \frac{\omega}{4}\Big\lfloor(\lambda_2 + 2\mu_2)(\epsilon_r\,\overline{\epsilon}_r + \epsilon_\theta\,\overline{\epsilon}_\theta) + \lambda_2(\epsilon_r\,\overline{\epsilon}_\theta + \epsilon_\theta\,\overline{\epsilon}_r) + 4\mu_2\,\epsilon_{r\theta}\,\overline{\epsilon}_{r\theta}\Big\rfloor$$

This can easily be shown by using the fact that $\left|\epsilon_{kk}\right|^2$ and $\left|\epsilon_{ij}\right|^2$ are invariant under the transformation of coordinates and become

$$\left|\epsilon_{kk}\right|^2 \Longrightarrow (\epsilon_r + \epsilon_\theta)(\overline{\epsilon}_r + \overline{\epsilon}_\theta)$$

$$\left|\epsilon_{ij}\right|^2 \Longrightarrow \epsilon_r\,\overline{\epsilon}_r + \epsilon_\theta\,\overline{\epsilon}_\theta + 2\epsilon_{r\theta}\,\overline{\epsilon}_{r\theta}$$

VARIATIONAL PRINCIPLES

1. Variational Principles in Three Dimensions

The field equations (6) and (10) for the coupled thermomechanical problem together with the boundary conditions are equivalent to two variational principles.

The variational principle for the equations of motion and the constitutive equations (2) and (3) can be stated as: of all displacement functions u_i satisfying prescribed displacements u_i on A_u, the displacement function satisfying the equations of motion (2), the constitutive equations (3) and the traction boundary condition on A_σ is determined by

$$\delta_u \left\{ \int_V (U_v - K_v)\, dV - \int_{A_\sigma} u_i \, \overset{\circ}{\sigma}_{ij} \, n_j \, dA \right\} = 0 \tag{15}$$

where U_v and K_v are analogous to the elastic strain energy density and kinetic energy density and are given by

$$U_v = \frac{\lambda^*}{2}(u_{k,k})^2 + \frac{\mu^*}{2}(u_{i,j})(u_{i,j} + u_{j,i})$$

$$K_v = \frac{1}{2} \rho \, \omega^2 \, u_i \, u_i$$

$\overset{\circ}{\sigma}_{ij} \, n_j$ = prescribed traction on A_σ

n_j = direction cosines of the outward unit normal to the surface A_σ

δ_u means that the variations must be taken with respect to the displacement function only.

If the kinematic equations (10) are also satisfied (i.e., we define strain functions to satisfy Eq. (10)), we can write

$$U_v = \frac{\lambda^*}{2}(\epsilon_{kk})^2 + \mu^* \epsilon_{ij} \epsilon_{ij}$$

This variational principle is analogous to Hamilton's principle in dynamic elasticity.

Comparing Eq. (15) with the variational principle given by Schapery in [7] we see that here we can choose trial functions for the displacement which need only satisfy the displacement boundary conditions of the problem whereas in [7] Schapery must choose displacement and stress functions which must already satisfy the equations of motion. The latter principle thus appears more restrictive and would be more difficult to apply in complicated problems.

The variational principle can be proved by carrying out variations with respect to the function u_i to yield

$$- \int_V \left\{(\lambda^* u_{k,k})_{,i} + (\mu^* u_{i,j})_{,j} + (\mu^* u_{j,i})_{,j} + \rho\, \omega^2 u_i\right\} \delta u_i \, dV$$

$$+ \int_{A_\sigma} \left\{\lambda^* u_{k,k} \delta_{ij} + \mu^*(u_{i,j} + u_{j,i}) - \overset{\circ}{\sigma}_{ij}\right\} n_j \, \delta u_i \, dA = 0 \qquad (16)$$

In view of the arbitrariness of δu_i, this expression equals zero only if the equations of motion (2), the constitutive equations (3), and the traction boundary conditions on A_σ are satisfied.

If we restrict the admissible class of displacement functions such that the boundary conditions for both the displacements on A_u and traction on A_σ (through Eq. (3)) are satisfied, the surface integral drops out of Eq. (16) and we are left with a simplified form of the principle

$$\int_V \left\{ (\lambda^* u_{k,k})_{,i} + (\mu^* u_{i,j})_{,j} + (\mu^* u_{j,i})_{,j} + \rho\, \omega^2 u_i \right\} \delta u_i \, dV = 0 \tag{17}$$

Equation (16) (or (17) which is a special case of (16)) can be considered to be an alternative form of the variational principle (15). Equation (17) can be considered to be a Galerkin formulation of the problem.

It is useful to compare the relative advantages of the two forms. Equation (15) uses energy invariants and therefore appears more convenient in complicated coordinate systems. However, when carrying out a Rayleigh-Ritz method of solution, use of Eq. (16) can save a large amount of calculations since the variations have already been carried out.

It must be remembered that λ^* and μ^* are temperature dependent and in order to get the temperature field we require another variational principle from the energy equation. This can be stated as follows: of all temperature distributions which satisfy prescribed $\mathring{T}$ on A_T, the temperature distribution which also satisfies the energy equation (7) and the heat flow boundary condition on A_H is determined by

$$\delta_T \left\{ \int_V (S_T - S_M)\, dV + \int_{A_H} \mathring{H}\, T\, dA \right\} = 0 \tag{18}$$

where S_T is proportional to the entropy production density resulting from temperature gradients (see [7])

$$S_T = \frac{1}{2} KT,_i \, T,_i$$

and S_M is the integral of the mechanical dissipation

$$S_M = 2 \int^{T} D \, dT'$$

$\overset{\circ}{H}$ = prescribed heat flux per unit area out of the body.

δ_T means the variations must be taken with respect to the temperature only;

D is as given by Eqs. (7) and (10).

This principle can be proved by taking variations with respect to T to yield

$$- \int_V (2D + KT,_{ii}) \, \delta T \, dV + \int_{A_H} (K \, T,_i \, n_i + \overset{\circ}{H}) \, \delta T \, dA = 0 \qquad (19)$$

In view of the arbitrariness of δT, this expression is zero only if the energy equation (10) and the heat flow boundary condition on A_H is satisfied.

If we restrict the admissible class of temperature functions such that the boundary conditions for both the temperature on A_T and heat flow on A_H are satisfied, the surface integral drops out of Eq. (19) and we are left with

$$\int_V (2D + KT,_{ii}) \, \delta T \, dV = 0 \qquad (20)$$

Equation (19) is an alternative form of the variational principle (18). Equation (18) uses thermodynamic invariants and the comments made earlier about the two forms of the variational principle for the equations of motion apply here too.

The variational principles for displacement and temperature (Eqs. (15) and (18)) are entirely equivalent to the field equations (2), (3), and (10), with their associated boundary conditions. The displacement and temperature functions can be obtained by simultaneously making the appropriate integrals stationary with respect to displacement and temperature respectively. The first equation (15) could be regarded as getting a stationary "cost" function, and the second (18) as a constraint, or vice versa.

A Rayleigh-Ritz procedure can be used to obtain the displacement (and hence the stress components) and temperature distribution. This is done in a one-dimensional example presented later in the report.

2. Variational Principles in Two-Dimensional Polar Coordinates for Plane Strain

The variational principle for displacement (Eq. (15)) takes the form: of all displacement and strain functions satisfying prescribed displacements on A_u and the kinematic Eqs. (13), the displacement function satisfying the equations of motion (11), the constitutive equations (12), and the traction boundary conditions on A_σ are given by

$$\delta_u \left\{ \int_V \left\lfloor \frac{\lambda^*}{2}(\epsilon_r + \epsilon_\theta)^2 + \mu^*(\epsilon_r^2 + \epsilon_\theta^2 + 2\epsilon_{r\theta}^2) - \frac{\rho\omega^2}{2}(u_r^2 + u_\theta^2) \right\rfloor dV - \int_{A_\sigma} \left\lfloor (n_r \mathring{\sigma}_r + n_\theta \mathring{\sigma}_{r\theta}) u_r + (n_r \mathring{\sigma}_{r\theta} + n_\theta \mathring{\sigma}_\theta) u_\theta \right\rfloor dA \right\} = 0 \quad (21)$$

where

$$\underset{\sim}{n} = n_r \underset{\sim}{e}_r + n_\theta \underset{\sim}{e}_\theta$$

$\underset{\sim}{n}$ being the outward normal to A_σ and $\underset{\sim}{e}_r$, $\underset{\sim}{e}_\theta$ the unit vectors in the radial and tangential directions.

Equation (21) is obtained from Eq. (15) by a transformation of coordinates using the fact that ϵ_{kk} and $\epsilon_{ij}\,\epsilon_{ij}$ are invariants.

The alternative form of Eq. (21), for the case (for simplicity) where the normal $\underset{\sim}{n} = \underset{\sim}{e}_r$ (i.e. for a circular boundary) is obtained by taking variations with respect to u_r and u_θ

$$- \int_V \Big[\Big\{ \frac{\partial}{\partial r}(\lambda^*(\epsilon_r + \epsilon_\theta) + 2\mu^*\, \epsilon_r) + \frac{2}{r}\frac{\partial}{\partial\theta}(\mu^*\, \epsilon_{r\theta}) + \frac{2\mu^*}{r}(\epsilon_r - \epsilon_\theta)$$

$$+ \rho\, \omega^2\, u_r \Big\}\, \delta u_r + \Big\{ \frac{1}{r}\frac{\partial}{\partial\theta}\, (\lambda^*(\epsilon_r + \epsilon_\theta) + 2\mu^*\, \epsilon_\theta) + 2\,\frac{\partial}{\partial r}(\mu^*\, \epsilon_{r\theta})$$

$$+ \frac{4\mu^*\, \epsilon_{r\theta}}{r} + \rho\, \omega^2\, u_\theta \Big\}\, \delta u_\theta \Big]\, dV + \int_{A_\sigma} \Big[\Big\{ \lambda^*(\epsilon_r + \epsilon_\theta) + 2\mu^*\, \epsilon_r - \mathring{\sigma}_r \Big\} \delta u_r$$

$$+ \Big\{ 2\mu^*\, \epsilon_{r\theta} - \mathring{\sigma}_{r\theta} \Big\}\, \delta u_\theta \Big]\, dA = 0 \qquad (22)$$

where $\mathring{\sigma}_r$ and $\mathring{\sigma}_{r\theta}$ are the prescribed traction components on A_σ .

This expression is seen to vanish when the Eqs. (11), (12), and the traction boundary conditions on A_σ are satisfied.

If the normal points inwards (i.e. $\underset{\sim}{n} = -\underset{\sim}{e}_r$) the sign of the surface integral changes.

Once again, if the trial functions for displacement components are chosen such that the traction boundary conditions are satisfied on A_σ the surface integral vanishes and we have a simpler form of the variational principle.

The variational principle for the energy equation (18) becomes

$$\delta_T \left\{ \int_V \left[\frac{K}{2} \left\{ \left(\frac{\partial T}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial T}{\partial \theta}\right)^2 \right\} - 2 \int^T D \, dT' \right] dV + \int_{A_H} \mathring{H} \, T \, dA \right\} = 0 \quad (23)$$

where D is given explicitly in Eq. (14).

Its alternative form, obtained as usual by taking variations with respect to the temperature, for the case $\underset{\sim}{n} = \underset{\sim}{e}_r$ is

$$- \int_V (K \nabla^2 T + 2D) \, \delta T \, dV + \int_{A_H} (\mathring{H} + K \frac{\partial T}{\partial r}) \, \delta T \, dA = 0 \qquad (24)$$

where ∇^2 is as given in Eq. (14).

Once again, as expected, this expression vanishes when the energy equation (14) and the heat flow boundary condition on A_H is satisfied.

AN EXAMPLE IN ONE DIMENSION

1. The Problem

The problem of steady state longitudinal waves in a viscoelastic rod with thermomechanical coupling is now solved using the one dimensional versions of the variational principles presented in the previous section. A Rayleigh-Ritz procedure is used on the alternative forms of the variational principles. The same problem, including time dependence, was solved by Huang and Lee [9] using a finite difference approach. The results obtained here are compared with some steady state results given in [9]. The question of how convergence to the solution given in [9] depends on the number of coordinate functions used is discussed.

Let us consider a viscoelastic rod of length ℓ insulated on its lateral surface as shown in Fig. 1. The left end is free while the right end is vibrated at a frequency ω with a constant stress amplitude σ_0 (real), so that the prescribed stress at this end is $\sigma_0 \cos \omega t$. The temperature of the vibrator is assumed constant at T_0 while a radiation boundary condition is assumed at $x = 0$.

The boundary conditions can therefore be written as

$$\begin{aligned} x = 0 \qquad & \sigma = 0 \\ & \frac{dT}{dx} = c(T - T_0) \\ x = \ell \qquad & \sigma = \sigma_0 \\ & T = T_0 \end{aligned} \tag{25}$$

where $c = h/K$ is the ratio of the surface conductance h to the thermal conductivity K of the viscoelastic material. Note that the problems of uniform normal or shear traction on the surface of a wide slab with the stated thermal boundary conditions prescribed on the slab surface are mathematically equivalent problems. Note also that here we have mixed thermal boundary conditions but this can be taken care of in the variational principle as shown later.

2. The Field Equations and Variational Principles

The equations of motion (2) and the constitutive equations (3) reduce to

$$\frac{d}{dx}\left(E^* \frac{du}{dx}\right) + \rho\, \omega^2 u = 0 \tag{26}$$

$$\sigma = \sigma_1 + i\, \sigma_2 = E^* \frac{du}{dx} \tag{27}$$

where $E^* = E_1 + i\,E_2$ is the complex Young's modulus which is a function of the temperature through the reduced frequency (see Eq. (5)).

The steady state energy equation becomes

$$K \frac{d^2T}{dx^2} + \frac{\omega}{2} E_2 \left|\frac{du}{dx}\right|^2 = 0 \tag{28}$$

where, as before, $\left|\frac{du}{dx}\right|^2 = \frac{du}{dx}\frac{d\bar{u}}{dx}$.

Note that for the one-dimensional strain problem, E^* must be replaced by $\lambda^* + 2\mu^*$ and E_2 by $\lambda_2 + 2\mu_2$.

The corresponding variational principle for displacement becomes: of all possible displacement functions, the one satisfying the equation of motion (26), the constitutive equation (27) and the stress boundary conditions from Eq. (25) is determined from

$$\delta_u \left\{ \int_0^{\ell} \left[-\frac{E^*}{2}\left(\frac{du}{dx}\right)^2 + \frac{\rho\,\omega^2\,u^2}{2} \right] dx + \sigma_0\,u(\ell) \right\} = 0 \tag{29}$$

If the admissible class of displacement functions is restricted such that the stress boundary conditions are already satisfied (through Eq. (27)), the alternative form of the variational principle takes the simplified form

$$\int_0^{\ell} \left\{ \frac{d}{dx}\left(E^* \frac{du}{dx}\right) + \rho\,\omega^2\,u \right\} \delta u\,dx = C \tag{30}$$

The temperature variational principle takes the form: of all temperature distributions which satisfy $T(\ell) = T_0$, the temperature distribution which also satisfies the energy equation (28) and the radiation boundary condition at $x = 0$ (see Eq. (25)) is determined from

$$\delta_T \left\{ \int_0^{\ell} \left(\frac{1}{2} K \left(\frac{dT}{dx}\right)^2 - \int^{T} \frac{\omega\, E_2(T')}{2} \left|\frac{du}{dx}\right|^2 dT' \right) dx + h\left(\frac{T^2}{2} - T\, T_0\right)_{x=0} \right\} = 0 \tag{31}$$

Taking variations and integrating by parts, the alternative form is obtained as

$$- \int_0^{\ell} \left(K \frac{d^2T}{dx^2} + \frac{\omega\, E_2}{2} \left|\frac{du}{dx}\right|^2 \right) \delta T\, dx + \left\{ \left(h(T - T_0) - K \frac{dT}{dx} \right) \delta T \right\}_{x=0} = 0 \tag{32}$$

which is true only if Eq. (28) and the radiation boundary condition at $x = 0$ is satisfied.

As before, if the temperature is chosen such that both the temperature and radiation boundary conditions (at $x = \ell$ and at $x = 0$) are already satisfied, the temperature must be determined from

$$\int_0^{\ell} \left(K \frac{d^2T}{dx^2} + \frac{\omega\, E_2}{2} \left|\frac{du}{dx}\right|^2 \right) \delta T\, dx = 0 \tag{33}$$

Equations (30) and (33) are used in further calculations in this section. The object is to find the spatial distribution of temperature, displacement and then stress.

3. The Properties of the Material

A Lockheed solid propellant is an example of a thermorheologically simple material in which, within a wide reduced frequency range, the complex shear compliance $J^* = J_1 - i\,J_2$ can be represented by (see [9])

$$J_1 = k_1(\omega\, a_T)^{-n_1}$$

$$J_2 = k_2(\omega\, a_T)^{-n_1}$$

where

$$\frac{k_2}{k_1} = \tan\left(\frac{n_1\,\pi}{2}\right)$$

$$(a_T)^{n_2} = \frac{T_{n_2}}{T - T_1}$$

and k_1 , k_2 , n_1 , n_2 , T_1 and T_{n_2} are constants. The tensile compliance $D^* = 1/E^*$ is related to the shear compliance J^* and the bulk compliance B^* by the equation

$$D^* = \frac{J^*}{3} + \frac{B^*}{9}$$

and whenever J^* is greater than B^* by at least two orders of magnitude, we can write

$$D^* = D_1 - i\,D_2 = \frac{J^*}{3}$$

$$D_1 = c_1\,\omega^{\beta}(T - T_1)^{\gamma} \tag{34}$$

$$D_2 = c_2\,\omega^{\beta}(T - T_1)^{\gamma}$$

where c_1 , c_2 , β , γ are constants.

E_1 and E_2 are now obtained from above as

$$E_1 = \frac{D_1}{D_1^2 + D_2^2}$$
$$E_2 = \frac{D_2}{D_1^2 + D_2^2} \tag{35}$$

4. Method of Solution

The Rayleigh-Ritz procedure [11] is now used to obtain approximate solutions for the temperature and displacement (and then stress) functions from the variational equations (30) and (33).

The following dimensionless quantities are used

$$q = \frac{x}{\ell} \; , \quad \tau = \frac{T - T_1}{T_0 - T_1} \; , \quad \varkappa = c\,\ell \tag{36}$$

Equations (30) amd (33) are nonlinear and it is not possible to choose an orthogonal set of coordinate functions for the displacement and temperature. For convenience, it is assumed that the displacement and temperature distributions can be approximated by a linear combination of polynomials with coefficients to be determined. These functions are chosen such that they satisfy the boundary conditions (Eq. (25)) for all choices of these unknown coefficients. Also, the number of terms in the series are parameters which can be set in the resulting algebraic equations for the coefficients. This enables comparison of successive approximations with the solution in [9] and thus an estimate of the degree of accuracy as a function of the number of terms taken is obtained.

The non-dimensional temperature is written as

$$\tau(q) = 1 + (1 - q) \left\{ b_0 + e_1 b_0 q + q^2 \sum_{i=2}^{m} b_i (1 - q)^{i-2} \right\} \tag{37}$$

where $e_1 = \varkappa + 1$, m is a parameter and b_0 , b_2 , b_3 ... b_m are m real constants that are to be determined ($m < 2$ implies $\sum_{i=2}^{m} \equiv 0$).

It is easily seen that

$$\left.\frac{d\tau}{dq}\right|_{q=0} = b_0(e_1 - 1) = \varkappa\, b_0 = \varkappa(\tau - 1)_{q=0}$$

and

$$\tau\Big|_{q=1} = 1$$

which means that the thermal boundary conditions from Eq. (25) are satisfied in terms of the non-dimensional variables defined in Eq. (36).

The complex strain is written as

$$\epsilon = \epsilon_1 + i\, \epsilon_2 = \frac{du}{dx} = q \sum_{i=0}^{n} a_i (1 - q)^i \tag{38}$$

where

$$a_0 = a_0^R + i\, a_0^I = D^*\Big|_{q=1} \sigma_0 = (c_1 - i\, c_2)\, \omega^\beta (T_0 - T_1)^\gamma \sigma_0$$

n is a parameter and a_1 , a_2 , a_3 ... a_n are n complex constants that are to be determined.

As before, it is obvious that with this choice of strain

$$\sigma\Big|_{q=0} = 0 \quad , \quad \sigma\Big|_{q=1} = E^*\Big|_{q=1} a_0 = \sigma_0$$

so that the stress boundary conditions from Eq. (25) are satisfied.

Writing $a_k = a_k^R + i\, a_k^I$ $(k = 1,n)$ this choice of functions leads to $(2n + m)$ real unknowns which must be determined from an equal number of algebraic equations.

These nonlinear algebraic equations are now determined from the variational equations (30) and (33). Substituting the displacement and temperature expressions into Eq. (30), carrying out the necessary integrations and equating the coefficients of $\delta\, a_j^R$ $(j = 1,n)$ to zero gives

for $j = 1,2,3,\ldots n$

$$\frac{2}{(j+1)(j+2)(j+3)} + \sum_{k=0}^{n} a_k^R\, d[f(j,k) - f(j,k+1)]$$
$$- \sum_{k=0}^{n} \frac{(a_k^R\, a_0^R + a_k^I\, a_0^I)}{|a_0|^2} (I_{k+j} - 2I_{k+j+1} + I_{k+j+2}) = 0 \tag{39}$$

and equating the coefficients of $\delta\, a_j^I$ $(j = 1,n)$ to zero gives

for $j = i,2,3,\ldots n$

$$\sum_{k=0}^{n} a_k^I\, d[f(j,k) - f(j,k+1)]$$
$$- \sum_{k=0}^{n} \frac{(a_k^I\, a_0^R - a_k^R\, a_0^I)}{|a_0|^2} (I_{k+j} - 2I_{k+j+1} + I_{k+j+2}) = 0 \tag{40}$$

where I_p (p an integer) is a nonlinear function of e_1 , b_0 , b_2 , b_3 , ..., b_m defined as

$$I_p = \int_0^1 \frac{(1-q)^p \, dq}{\left\{1 + (1-q)\left[b_0 + e_1 b_0 q + q^2 \sum_{i=2}^{m} b_i (1-q)^{i-2}\right]\right\}^{\gamma}}$$

$f(j,k)$ is a function of integers

$$f(j,k) = \frac{k(k+j+4) + (k+2)(j+3)}{(j+2)(j+3)(k+1)(k+2)(k+j+3)(k+j+4)}$$

d is a non-dimensional parameter

$$d = \frac{\rho \, \omega^2 \, \ell^2}{\sigma_0}$$

and

$$|a_0|^2 = (a_0^R)^2 + (a_0^I)^2$$

Note that Eq. (30) requires the displacement u in addition to the strain. Integration of Eq. (38) leads to an extra constant, say c_0 , but also an extra equation obtained by equating the coefficient of δc_0 to zero. This extra constant c_0 has been eliminated from Eqs. (39) and (40) given above.

Next, substituting the displacement and temperature expressions into Eq. (33) and equating the coefficient of δb_0 to zero gives

$$\frac{(3 + e_1) e_1 b_0}{3} - \sum_{k=2}^{m} b_k (g_{k-1} - 2g_k + g_{k+1}) + V \sum_{i,k=0}^{n} \Big[(a_k^R a_i^R + a_k^I a_i^I) \times$$

$$\left\{(1+e_1) I_{i+k+1} - (2 + 3e_1) I_{i+k+2} + (1 + 3e_1) I_{i+k+3} - e_1 I_{i+k+4}\right\}\Big] = 0 \quad (41)$$

and equating the coefficients of δb_j $(j = 2,m)$ to zero gives

for $j = 2,3,4...m$

$$\frac{4e_1 b_0}{j(j+1)(j+2)} - \sum_{k=2}^{m} b_k[h(j,k-1) - 2h(j,k) + h(j,k+1)]$$

$$+ V \sum_{i,k=0}^{n} (a_k^R a_i^R + a_k^I a_i^I) \Big\{ I_{i+j+k-1} - 4I_{i+j+k} + 6I_{i+j+k+1}$$

$$- 4I_{i+j+k+2} + I_{i+j+k+3} \Big\} = 0 \tag{42}$$

where g_k and $h(j,k)$ are given by

$$g_k = \frac{(k+1+e_1)(k-1)}{(k+1)}$$

$$h(j,k) = \frac{2k(k-1)}{(k+j)(k+j-1)(k+j-2)}$$

V is the non-dimensional parameter

$$V = \frac{-\ell^2 c_2(T_0 - T_1)^{-\gamma-1} \omega^{1-\beta}}{2K(c_1^2 + c_2^2)}$$

and I_p has been defined before in Eq. (40).

Equations (39), (40), (41), and (42) constitute a set of $(2n+m)$ nonlinear algebraic equations for the $(2n+m)$ unknowns a_k^R $(k = 1,n)$, a_k^I $(k = 1,n)$, b_0 and b_k $(k = 2,m)$.

The integral I_p given in Eq. (40) can be approximately evaluated by expanding the denominator in a binomial series retaining only linear terms and carrying out the integration.

For example, for $m = 5$, we get

$$I_p = \gamma(1 + b_0)^{-\gamma-1} \sum_{k=0}^{6} \frac{w_k\ K!\ p!}{(p+k+1)\ !}$$

where

$$w_1 = -(e_1 - 1)\ b_0$$
$$w_2 = -(b_2 + b_3 + b_4 + b_5 - e_1\ b_0)$$
$$w_3 = (b_2 + 2b_3 + 3b_4 + 4b_5)$$
$$w_4 = -(b_3 + 3b_4 + 6b_5)$$
$$w_5 = (b_4 + 4b_5)$$
$$w_6 = -b_5$$

The values of I_p for other values of m can be easily calculated.

This approximation for I_p proved sufficiently accurate for the calculations carried out. I_p can, of course, be more accurately determined by numerical integration for each trial value of b_0, b_2, ... b_m during iterative solving of the nonlinear algebraic equations (39), (40), (41), and (42).

The stresses are determined from the strains and temperature from

$$\begin{aligned} \sigma_1 &= E_1\ \epsilon_1 - E_2\ \epsilon_2 \\ \sigma_2 &= E_2\ \epsilon_1 + E_1\ \epsilon_2 \end{aligned} \tag{43}$$

and the stress at any time

$$Re(\tilde{\sigma}(x,t)) = Re(\sigma\ e^{i\omega t}) = \sigma_1 \cos \omega t - \sigma_2 \sin \omega t \tag{44}$$

These equations follow immediately from Eq. (27). E_1 and E_2 are determined as functions of temperature from Eqs. (34) and (35).

Non-dimensional stresses s_1 and s_2 are defined as

$$s_1 = \lambda \sigma_1 \quad , \quad s_2 = \lambda \sigma_2$$

and at $x = \ell$

$$s_0 = \lambda \sigma_0$$

where

$$\lambda = [2K \omega \rho (T_0 - T_1)]^{-1/2} \qquad (45)$$

5. Results and Conclusions

Numerical calculations have been carried out for the following data for a Lockheed solid propellant [9] whose mechanical and thermal properties are qualitatively typical of many viscoelastic solids

$c_1 = 4.61 \times 10^{-11}$ (psi)$^{-1}$ (sec)$^{\beta}$ ($^\circ$F)$^{-\gamma}$

$c_2 = 1.62 \times 10^{-11}$ (psi)$^{-1}$ (sec)$^{\beta}$ ($^\circ$F)$^{-\gamma}$

$\beta = -0.214$ $\qquad \gamma = 3.21$

$\kappa = 1.0$ † $\qquad T_0 = 65^\circ$F

$T_1 = -125^\circ$F $\qquad \ell = 3$ in.

$\ell^2 \rho = 1.023 \times 10^{-4}$ psi-sec^2

$2K\rho(T_0 - T_1) = 8.08 \times 10^{-4}$ psi^2-sec

$\omega = 10^4$ rad/sec. $\qquad s_0 = 0.5$ ($\sigma_0 = 1.42$ psi)

† In [9] κ should read 1.0 instead of 0.1.

The nonlinear algebraic equations (39), (40), (41), and (42) were solved numerically in a computer for different values of n and m. The subroutine used is given in [14]. The method is a compromise between the Newton-Raphson algorithm and the method of steepest descent.

Figures 2, 3 and 4 show the resulting τ, s_1 and s_2 distributions for different values of n and m and also the solution from [9] obtained by the method of finite differences. The solution for $n = 1$, $m = 0$ is crude but we see that the convergence to the true solution is very rapid. Figures 5, 6, and 7 show the approximate solutions for $n = 4$, $m = 3$. Even with these relatively small number of terms, the stress solutions are practically identical to those given in [9], while the temperature solution is well within engineering accuracy. The algorithm for solving the nonlinear algebraic equations converges very quickly and more accurate solutions can be obtained, if desired, by taking larger values of n and m.

As mentioned earlier, if the steady state values of stress and temperature are of interest (this is often the case in design), the method used here, which yields the steady state directly, is superior to that used by Huang and Lee in [9] where the complete time histories of the above mentioned quantities were determined. In some cases in [9] the authors obtained the steady state solutions by numerically integrating forward in time till the variables of interest did not change significantly. In other cases, they did not integrate upto the steady state but stopped at some large value of time.

The results obtained are thus very satisfactory as long as s_1 and s_2 are sufficiently smooth functions so that approximation by a series of polynomials is efficient. The nature of the spatial distribution of stress depends upon the particular choice of frequency and driving stress. For a given driving frequency, larger driving stresses lead to larger temperatures since more mechanical energy is dissipated as heat. This causes the material to become softer, so that lower stress wave velocities and therefore lower wave lengths result. If s_1 and s_2 are rapidly oscillating functions of q, the polynomial series is no longer efficient since a larger number of terms must be taken to get the required accuracy and the lack of orthogonality of the polynomials gives rise to Hilbert matrices. This results in convergence problems for the algorithm used to solve the algebraic equations. The variational principles, however, should work fine for these cases, if, for example, trigonometric functions are chosen instead.

To sum up, the variational approach seems comparable to the finite difference approach for waves in one dimension and ought to be more efficient in two or three dimensions where the differential equations are partial and finite difference simulation becomes much more complicated. Solving for displacements instead of stresses has the advantage of automatic satisfaction of compatibility conditions and Mitchell's equations for multiply connected regions. The choice of coordinate functions is very important and must be made carefully.

REFERENCES

1. M. E. Gurtin, "Variational Principles in the Linear Theory of Viscoelasticity," Arch. Rational Mech. Anal., Vol. 13, 1963, p. 179.

2. M. J. Leitman, "Variational Principles in the Linear Dynamic Theory of Viscoelasticity," Technical Report No. 98, Division of Applied Mathematics, Brown University, 1965.

3. K. C. Valanis, "Exact and Variational Solutions to a General Visco-elasto-kinetic Problem," Journal of Applied Mechanics, Vol. 33, 1966, p. 888.

4. W. Kohn, J. A. Krumhansl and E. H. Lee, "Variational Methods for Dispersion Relations and Elastic Properties of Composite Materials," Paper No. 71-APMw-21. Forthcoming in the Journal of Applied Mechanics.

5. R. A. Schapery, "Applications of Thermodynamics to Thermomechanical, Fracture and Birefringent Phenomena in Viscoelastic Media," Journal of Applied Physics, Vol. 35, 1964, p. 1451.

6. R. A. Schapery, "Effect of Cyclic Loading on the Temperature in Viscoelastic Media with Variable Properties," AIAA Journal, Vol. 2, 1964, p. 827.

7. R. A. Schapery, "Thermomechanical Behaviour of Viscoelastic Media with Variable Properties Subjected to Cyclic Loading," Journal of Applied Mechanics, Vol. 32, 1965, p. 611.

8. M. A. Biot, "Variational Principles in Heat Transfer," Oxford Clarendon Press, 1970.

9. N. C. Huang and E. H. Lee, "Thermomechanical Coupling Behavior of Viscoelastic Solids Subjected to Cyclic Loading," Journal of Applied Mechanics, Vol. 34, 1967, p. 127.

10. R. Courant and D. Hilbert, "Methods of Mathematical Physics," Vol. 1, Interscience Publishers Inc., New York, 1966.

11. F. B. Hilderbrand, "Methods of Applied Mathematics," Prentice Hall Inc., New Jersey, 1965.

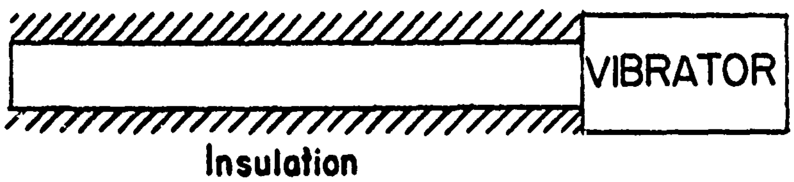
12. W. Flügge, "Viscoelasticity," Blaisdell Publishing Company, Waltham Massachusetts, 1967.

13. I. S. Sokolnikoff, "Mathematical Theory of Elasticity," McGraw Hill Book Company, Inc., New York, 1956.

14. M. J. D. Powell, "A Fortran Subroutine for Solving Systems of Non Linear Algebraic Equations," Report No. AERE-R.5947, Mathematical Branch, Theoretical Physics Division, A.E.R.E., Harwell, Berk., 1968.

ACKNOWLEDGEMENTS

The author is deeply indebted to Professor E. H. Lee of the Department of Applied Mechanics at Stanford University for suggesting the problem and for his very valuable advice and guidance during the course of this work. Sincere thanks are also expressed to Professor W. Yang, visiting professor at Stanford University, for many valuable suggestions during the computational phase of this work.



$x = 0$ $\qquad\qquad$ $x = \ell$

$\sigma = 0$ $\qquad\qquad$ $\mathrm{Re}(\tilde{\sigma}) = \sigma_0 \cos \omega t$

$\frac{dT}{dx} = c(T - T_0)$ $\qquad\qquad$ $T = T_0$

Figure 1. Boundary conditions for the one-dimensional problem.

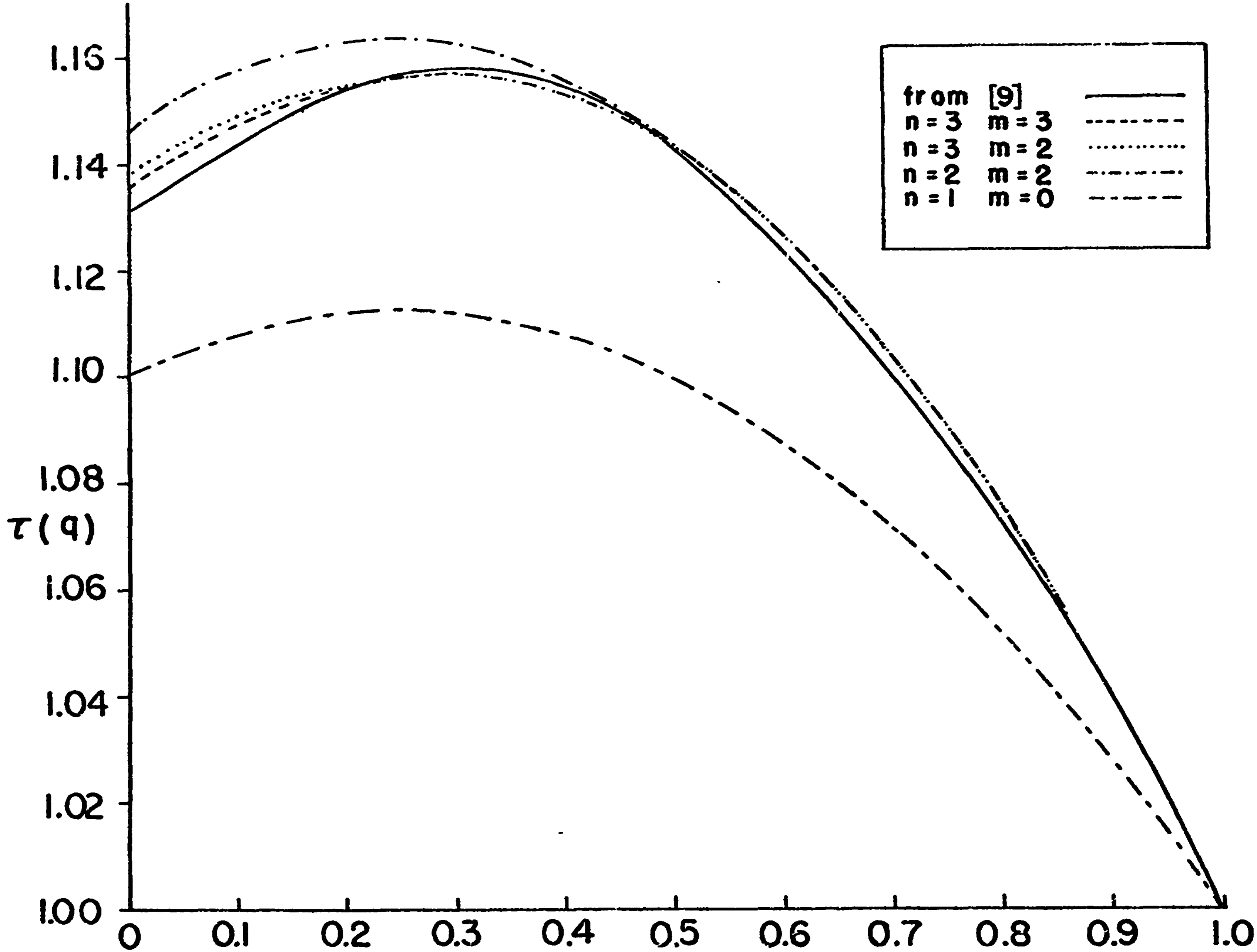


Figure 2. $\tau(q)$ curves for different values of n and m .

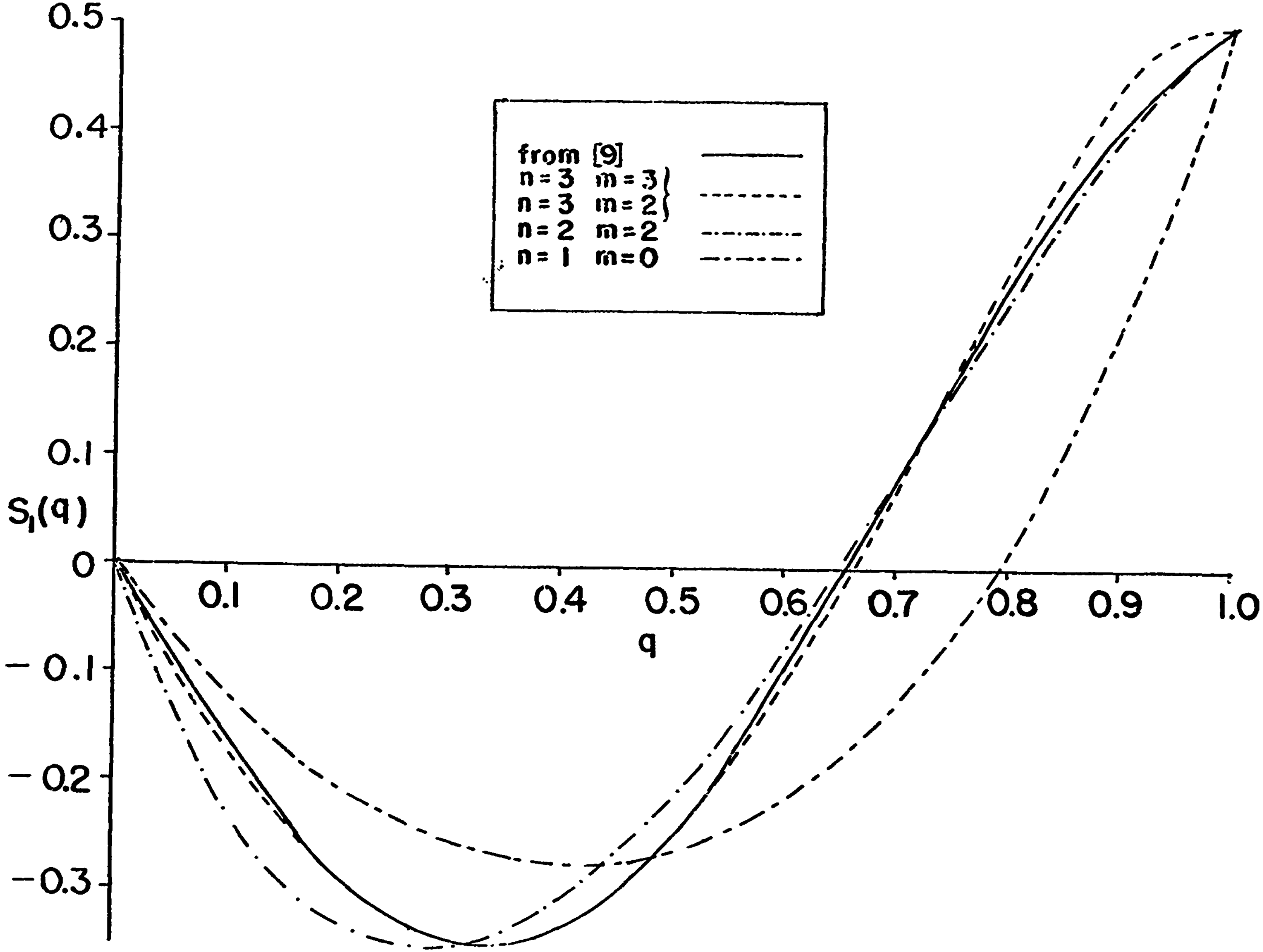


Figure 3. $S_1(q)$ curves for different values of n and m .

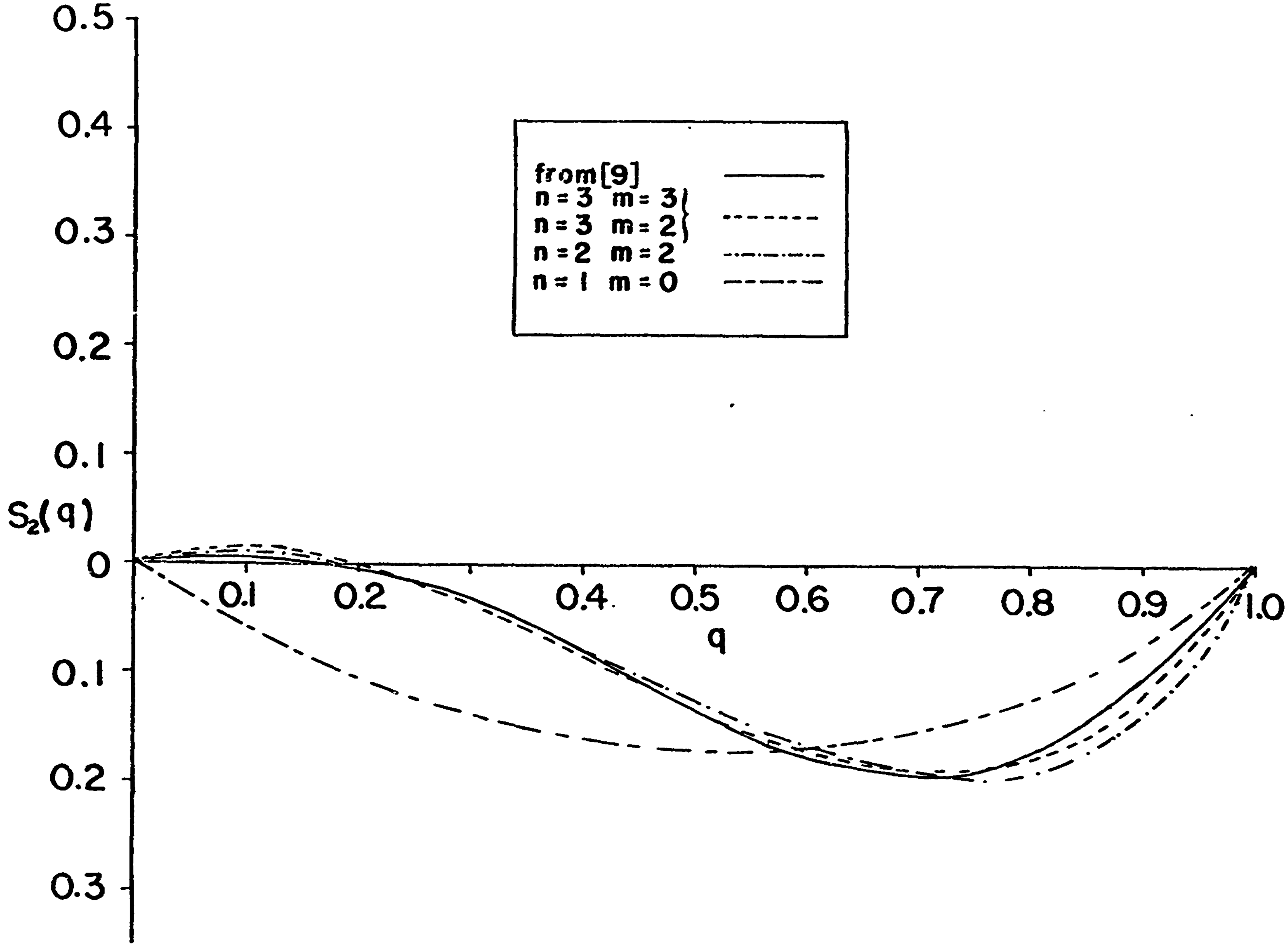


Figure 4. $S_2(q)$ curves for different values of n and m .

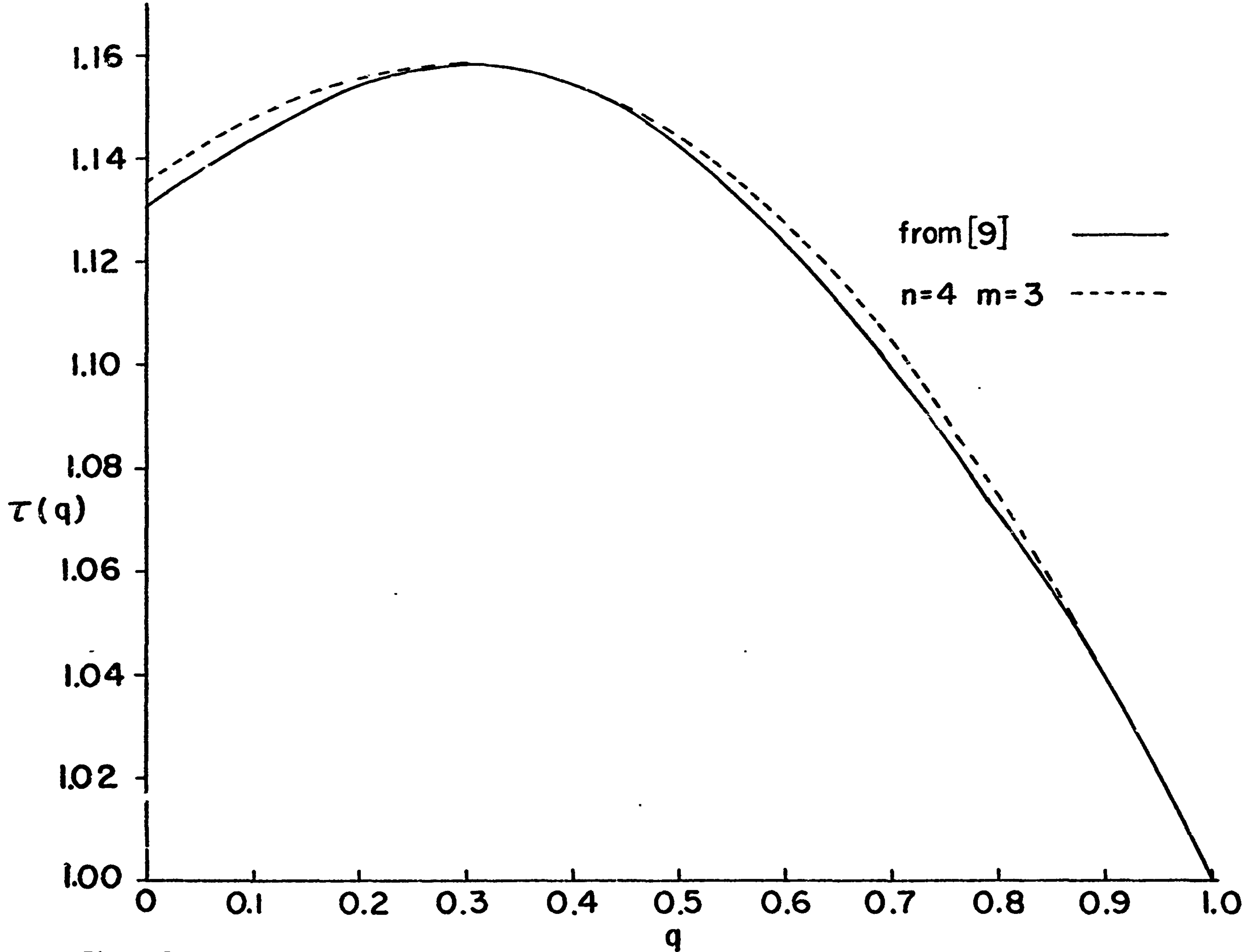


Figure 5. Comparison of $\tau(q)$ $(n = 4, m = 3)$ with correct solution.

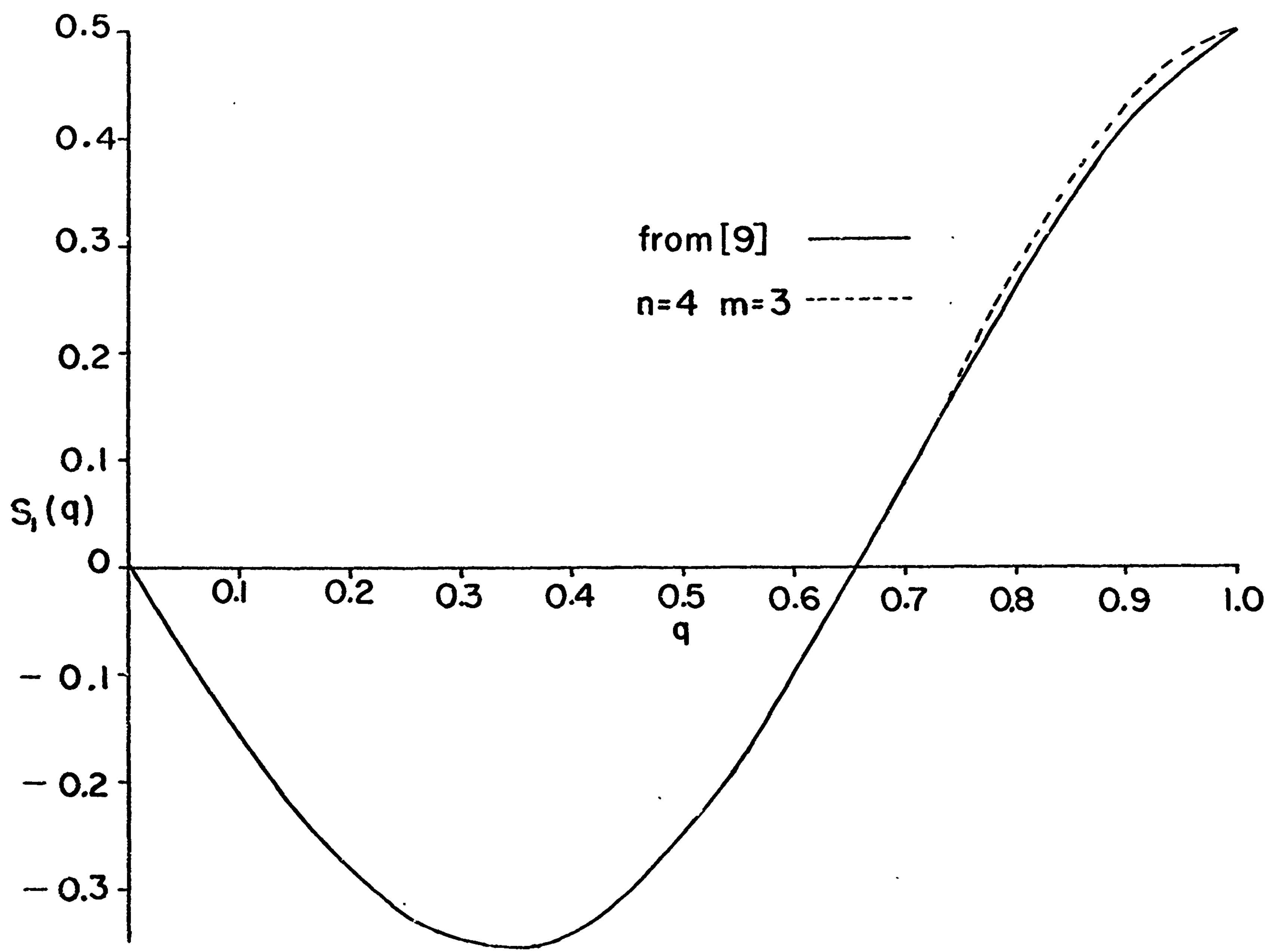


Figure 6. Comparison of $S_1(q)$ $(n = 4,\ m = 3)$ with correct solution.